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Translated by R.L.Z.

PMM U.S.S.R.,Vol.54,No.5,pp.637-640,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

TRIPLE-WAVE POTENTIAL FLOWS OF A POLYTROPIC GAS*

S.V. MELESHKO

A system of equations describing triple-wave solutions for unsteady isentropic potential flows of a polytropic gas was derived in /1/. A family of exact triple-wave solutions of the equations of gas dynamics with three arbitrary functions of one argument was constructed in /2/ for $1<\gamma<2$. Some applications and properties of this family were studied. In this paper we show that the triple-wave equations of /1/ are a system in involution and depend on one arbitrary function of two arguments.

1. The equations of motion of polytropic gas in the unsteady isentropic case can be written in the form

$$du/dt+\nabla\theta=0,\ d\theta/dt+\varkappa\theta\operatorname{div}\mathbf{u}=0 \qquad (1.1)$$
$$\theta=c^2/\varkappa,\ \varkappa=\gamma-1>0,\ d/dt=\partial/\partial t+u_\alpha\partial/\partial x_\alpha$$

*Prikl.Matem.Mekhan.,54,5,775-779,1990

where $\mathbf{u} = (u_1, u_2, u_3)'$ is the velocity vector, c the velocity of sound, γ is the polytropic exponent and summation from 1 to 3 is implied over repeating Greek indices. We consider triple-wave potential flows

$$\operatorname{rot} \mathbf{u} = 0 \tag{1.2}$$

Two possible cases should be considered: either the components of the velocity vector u_1, u_2, u_3 are functionally independent in some region D of the space x_1, x_2, x_3, t, and then we must take $\theta = \theta(u_1, u_2, u_3)$ in this region, or u_1, u_2, u_3 are functionally dependent in D (e.g., $u_3 = \Phi(u_1, u_2)$).

2. We will first consider the case of functionally independent u_1, u_2, u_3. Substituting $\theta = \theta(u_1, u_2, u_3)$ into the original Eqs.(1.1) and using the potential flow condition (1.2), we obtain an undetermined system of quasilinear differential equations

$$\begin{gathered}
\mathbf{S} = (S_1, S_2, S_3)' \equiv E\mathbf{u}_{,0} + G_\alpha \mathbf{u}_{,\alpha} = 0 \\
f_1 \equiv u_{1,2} - u_{2,1} = 0,\ f_2 \equiv u_{2,3} - u_{3,2} = 0,\ f_3 \equiv u_{1,3} - u_{3,1} = 0 \\
f_4 \equiv \psi_\alpha u_{\alpha,\alpha} + 2\theta_1\theta_2 u_{2,1} + 2\theta_2\theta_3 u_{3,2} + 2\theta_1\theta_3 u_{3,1} = 0 \\
\theta_i \equiv \partial\theta/\partial u_i,\ \psi_i \equiv \theta_i^2 - \varkappa\theta,\ G_i \equiv \Delta_i E,\ \Delta_i \equiv u_i + \theta_i \\
u_{i,j} = \partial u_i/\partial x_j,\ x_0 \equiv t,\ i = 1, 2, 3;\ j = 0, 1, 2, 3
\end{gathered} \tag{2.1}$$

where E is the 3x3 identity matrix. Without loss of generality, we will assume $\psi_3 \neq 0$ (this is accomplished by a rotation of the coordinate axes).

Consistency analysis of the overdetermined system of differential Eqs.(2.1) leads to the conclusion that the general solution of this system contains at most two arbitrary functions of two arguments. We first have to elucidate the question of algebraic independence with respect to high-order (second-order) derivatives of part of the extended equations and express the remaining equations in terms of the former equations.

From the prolongations $D_i\mathbf{S} = 0, D_i\Phi = 0, D_0\mathbf{S} = 0$ (D_k are total derivatives with respect to x_k, $k = 0, 1, 2, 3$) we determine the derivatives $\mathbf{u}_{,i3}$, $\mathbf{u}_{,0i}$, ($\mathbf{u}_{,ki} \equiv \partial^2 u/\partial x_i \partial x_k$, $i = 1, 2, 3$), and then substitute these derivatives into the remaining extended equations of system (2.1): $D_0\Phi = 0$, $D_k f_1 = 0$ ($k = 0, 1, 2, 3$). We then obtain

$$\begin{array}{cccl}
\mathbf{u}_{,11} & \mathbf{u}_{,12} & \mathbf{u}_{,22} & \\
0 & 0 & 0 & D_0\Phi + \Delta_\alpha D_\alpha \Phi - \Phi_\alpha D_\alpha \mathbf{S} \\
H_1 & H_2 & 0 & D_1 f_1 \\
0 & H_1 & H_2 & D_2 f_1 \\
M_1\Phi_1 & M_1\Phi_2 + M_2\Phi_1 & M_2\Phi_2 & -D_3 f_1 + M_\alpha D_\alpha \Phi \\
0 & 0 & 0 & D_0 f_1 + \Delta_\alpha D_\alpha f_1 - H_\alpha D_\alpha \mathbf{S}
\end{array} \tag{2.2}$$

$$(\Phi \equiv (f_2, f_3, f_4)',\ \Phi_i \equiv \partial\Phi/\partial u_{,i},\ H_i \equiv \partial f_1/\partial u_{,i} \quad M_i = H_i\Phi_3^{-1},\ i = 1, 2, 3)$$

Here the first row identifies the variables for which the coefficients are given in the following rows, and the last column represents the left-hand sides of the equations from which these coefficients are taken.

From the form of the matrices M_i and Φ_i it follows that in this case

$$M_i\Phi_j + M_j\Phi_i = 0,\ (i, j = 1, 2) \tag{2.3}$$

The solution of system (2.1) should thus satisfy some additional first-order equations:

$$D_0\Phi + \Delta_\alpha D_\alpha \Phi - \Phi_\alpha D_\alpha \mathbf{S} = 0 \tag{2.4}$$

$$D_3 f_1 - M_\alpha D_\alpha \Phi = 0 \tag{2.5}$$

$$D_0 f_1 + \Delta_\alpha D_\alpha f_1 - H_\alpha D_\alpha \mathbf{S} = 0 \tag{2.6}$$

These equations must be considered simultaneously with the rest of the system.

Of the equations (2.4)-(2.6) only the third equation in (2.4) does not identically vanish in view of system (2.1):

$$D_0 f_4 + \Delta_\alpha D_\alpha f_4 - \psi_\alpha D_\alpha S_\alpha - 2\theta_1\theta_2 D_1 S_2 - 2\theta_1\theta_3 D_1 S_3 - 2\theta_2\theta_3 D_2 S_3 = 0 \tag{2.7}$$

Using system (2.1) to eliminate the derivatives $u_{,3}$ and $u_{1,2}$ from (2.7), we obtain a quadratic form (relative to the derivatives $u_{i,j}$ ($i, j = 1, 2, 3;\ j \leqslant i$) $f_5 = 0$, whose coefficients are expressible in terms of θ, θ_i, θ_{ij}, where $\theta_{ij} = \partial^2\theta/\partial u_i \partial u_j$. For instance, the coefficient of $u_{3,2}^2$ equals $\psi_3 M_{23} (M_{ij} \equiv \psi_i (1+\theta_{jj}) + \psi_j (1+\theta_{ii}) - 2\theta_i\theta_j\theta_{ij})$, the form of all other coefficients is quite cumbersome.

If $M_{23} = 0$ for any rotation of the coordinate axes, then we must have (the missing equalities are obtained by a cyclic permutation of the indices)

$$\psi_1\theta_{23} + \theta_2\theta_3 (1+\theta_{11}) - \theta_1\theta_2\theta_{13} - \theta_1\theta_3\theta_{12} = 0 \ (1\ 2\ 3)$$
$$M_{12} = 0,\ M_{13} = 0,\ M_{23} = 0 \tag{2.8}$$

The last system of equations is linear and homogeneous with respect to θ_{ij}, $(1+\theta_{ii})$ $(i, j = 1, 2, 3;\ i \neq j)$ and its determinant equals $2(\varkappa\theta)^4(\theta_\alpha\theta_\alpha - \varkappa\theta)^2$. Therefore, if $\theta_\alpha\theta_\alpha - \varkappa\theta \neq 0$, then $\theta_{ij} = 0$, $\theta_{ii} = -1$ $(i, j = 1, 2, 3;\ i \neq j)$. Hence

$$\theta = c_0 - \frac{1}{2}\sum_{i=1}^{3} (u_i + c_i)^2 \quad (c_i = \text{const}) \tag{2.9}$$

Here $f_5 \equiv 0$, system (2.1) is in involution and its general solution contains two arbitrary functions of two arguments. It should be noted, however, that the change of variables $x_i' = x_i + c_i t$ reduces representation (2.9) to the Bernoulli integral, and the gas motion described by the system of Eqs.(2.1) corresponds to the general case of three-dimensional steady potential flows.

If $\theta_\alpha\theta_\alpha - \varkappa\theta = 0$, then from (2.8) it follows that $\gamma + 2 = 0$, which contradicts the condition $\varkappa > 0$. Thus, without loss of generality we may take $M_{23} \neq 0$.

Repeating the same procedure as in the derivation of (1.2)-(1.4), with f_1 replaced by $f_1' = (f_1, f_5)'$, we conclude that Eqs.(2.2)-(2.4), (2.6) preserve their form (with f_1 replaced by $(f_1, f_5)'$). Therefore, only the following equation is new relative to Eqs.(2.4)-(2.6):

$$f_6 \equiv D_0 f_5 + \Delta_\alpha D_\alpha f_5 - a_{1\alpha} D_1 S_\alpha - a_{2\alpha} D_2 S_\alpha = 0$$
$$a_{ki} = \partial f_5 / \partial u_{i,k} \quad (i = 1, 2, 3;\ k = 1, 2) \tag{2.10}$$

Carrying out all the prolongations indicated in (2.10) and eliminating the derivatives $u_{,0}$ in the resulting expression, we obtain

$$f_6 \equiv u_{\beta,\alpha} (a_{1\alpha} D_1 \Delta_\beta + a_{2\alpha} D_2 \Delta_\beta) = 0 \tag{2.11}$$

Manual substitution of the remaining main derivatives of the system of Eqs.(2.1), (2.7) is very complicated. It is accordingly done by computer using the program given in /3/.

After these substitutions, Eq.(2.10) can be represented in the form $f_6 = Ag$ where the function A is expressed in terms of $\theta(u_1, u_2, u_3)$ and its derivatives and g is a homogeneous form of third order in the derivatives $u_{i,j}$ $(i, j = 1, 2, 3;\ j \leqslant i)$ (the expressions for A and g are quite complicated and are therefore omitted here).

On the other hand, from Eqs.(2.1), (2.7) and the conditions $M_{23} \neq 0$, we obtain the equality $g = \psi_3 M_{23}\Delta$ $(\Delta \equiv \partial(u_1, u_2, u_3)/\partial(x_1, x_2, x_3))$. Therefore, if $g = 0$, then $\Delta = 0$. But then Eqs.(2.1) lead to a contradiction with the functional independence of u_1, u_2, u_3. We thus must take

$$A = 0 \tag{2.12}$$

Remark 1. Eq.(2.12) is identical with one of the necessary conditions for the existence of a triple wave for such flows /1/.

Remark 2. For system (2.1), (2.7) to be in involution it is necessary and sufficient that there exist matrices Λ_i, Λ_{3i} $(i = 1, 2, 3)$ such that /4/

$$D_0 \begin{Vmatrix} \Phi \\ f_1' \end{Vmatrix} + \Lambda_\alpha D_\alpha \begin{Vmatrix} \Phi \\ f_1' \end{Vmatrix} - \begin{Vmatrix} \Phi_\alpha \\ H_\alpha' \end{Vmatrix} D_\alpha S = 0, \quad \Lambda_{3\alpha} D_\alpha \begin{Vmatrix} \Phi \\ f_1' \end{Vmatrix} = 0$$

The previous manipulations indicate that such matrices in this case are

$$\Lambda_i = \Delta_i \begin{Vmatrix} E & 0 \\ 0 & E_2 \end{Vmatrix}, \quad \Lambda_{3i} = \| M_i', -\delta_{i3} E_2 \| \quad (i = 1, 2, 3)$$

(E_2 is the 2x2 identity matrix and δ_{ij} is the Kronecker delta). The general solution contains one arbitrary function of two arguments.

Thus, (2.10) is a necessary and sufficient condition for the overdetermined system of differential Eqs.(2.1) to be in involution, and its general solution contains one arbitrary function of two arguments.

3. Consider the case of the functional dependence $u_3 = \Phi(u_1, u_2)$. Substituting the expression for u_3 into the system of equations of gas dynamics (1.1), (1.2), we obtain

$$S = (S_1, S_2, S_3)' \equiv \mathbf{v}_{,0} + G_1\mathbf{v}_{,1} + G_2\mathbf{v}_{,2} = 0 \tag{3.1}$$
$$f \equiv (f_1, f_2, f_3)' \equiv \mathbf{v}_{,3} - \Phi_1\mathbf{v}_{,1} - \Phi_2\mathbf{v}_{,2} = 0$$
$$f_4 \equiv u_{1,2} - u_{2,1} = 0$$
$$G_k = (u_k + \Phi\Phi_k)E + \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \xi_k & \zeta_k & 0 \end{Vmatrix}, \quad k = 1, 2$$
$$\xi_1 = \zeta_2 = \varkappa\theta(1 + \Phi_1^2), \ \xi_2 = 0, \ \zeta_1 = 2\varkappa\theta\Phi_1\Phi_2$$
$$\mathbf{v} = (u_1, u_2, \theta)', \ \Phi_i = \partial\Phi/\partial u_i$$
$$\Phi_{ij} = \partial^2\Phi/\partial u_i \partial u_j, \quad i, j = 1, 2$$

Transforming as in Sect.2, we obtain a new first-order equation

$$f_5 = \theta_{,2}{}^2\Phi_{22} + \theta_{,1}{}^2\Phi_{11} + 2\theta_{,1}\theta_{,2}\Phi_{12} + \varkappa\theta\psi(\theta_{,i}, u_{i,j}, \Phi, \Phi_i, \Phi_{ij}) = 0 \tag{3.2}$$

The function f_5 is a homogeneous quadratic form in the derivatives $\theta_{,i}, u_{i,j}$, and ψ is a linear function in $\theta_{,i}$ $(i, j = 1, 2)$.

The case $\sum_{i,j} \Phi_{ij}^2 = 0$ reduces to the general solution for plane flow. We therefore must take $\sum_{i,j} \Phi_{ij}^2 \neq 0$. Then, rotating the coordinate axes (x_1, x_2) we may ensure the condition $\Phi_{22} \neq 0$.

Analysing the first prolongation of system (3.1), (3.2) we conclude that consistency of the system of differential Eqs.(3.1), (3.2) requires yet another first-order equation:

$$f_6 \equiv D_3f_5 - (a_{1\alpha}D_1f_\alpha + a_{2\alpha}D_2f_\alpha) - \Phi_1D_1f_5 - \Phi_2D_2f_5 = 0$$
$$(a_{ij} = \partial f_5/\partial u_{i,j}, \ a_{i3} = \partial f_5/\partial\theta_{,i}; \ i, j = 1, 2)$$

As in Sect.2, the expression for f_6 was determined by computer. Substituting the main derivatives of the system of Eqs.(3.1), (3.2), we obtain $f_6 = (\Phi_{12}^2 - \Phi_{11}\Phi_{22})g = 0$ with some function g.

On the other hand, since we are considering a triple wave, the Jacobian $\Delta \equiv \partial(u_1, u_2, \theta)/\partial(x_1, x_2, t) \neq 0$ (otherwise u_1, u_2, and θ would be functionally dependent). Moreover, from the form of the function g it follows that $g = \Phi_{22}\Delta$. This in turn implies that $g \neq 0$ and

$$\Phi_{12}^2 - \Phi_{11}\Phi_{22} = 0 \tag{3.3}$$

Thus, (3.3) are necessary and sufficient conditions for the existence of a triple wave when $u_3 = \Phi(u_1, u_2)$. As in the previous case, we have $f_6 \equiv 0$, so that the system of Eqs.(3.1), (3.2) is in involution with the general solution containing one arbitrary function of two arguments.

Remarks 3. Conditions (2.10) and (3.3) were obtained in /1/ under the assumption $\Delta \neq 0$. Here we have shown that this assumption is a necessary and sufficient condition for the existence of triple waves. In addition to condition (2.10) (or (3.3)), two more equations on the location function were obtained in /1/ (by transforming the hodograph in Eqs.(2.1) or (3.1)). It was verified by computer that the equation $f_6 = 0$ is identically true under this transformation. We thus conclude that this system of two equations on the location function with a given function $\theta = \theta(u_1, u_2, u_3)$ (or $u_3 = \Phi(u_1, u_2)$) is also in involution and the general solution of the Cauchy problem has one arbitrary function of two arguments.

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Translated by Z.L.